

PURIFICATION OF IMPURE DENSITY OPERATORS AND THE RECOVERY OF ENTANGLEMENTS

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Abstract

The need to retain the relative phases in quantum mechanics implies an addition law parametrized by a phase of two density operators required for the purification of a density matrix. This is shown with quantum tomography and the Wigner function. Entanglement is determined in terms of phase dependent multiplication.

Quantum mechanics as traditionally formulated[1] involves three principles. The states of the system are normalized vectors in a Hilbert space, (selfadjoint) linear operators correspond to (real) dynamical variables; and the expectation value for any dynamical variable is bilinear (sesquilinear) in the state vectors. But the overall phase of the state vector is irrelevant in those computations

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involving only that one state. The state corresponds to a “ray” in Hilbert space $\{\psi : e^{i\alpha}\psi_0\}$.

Alternatively the expectation value can be expressed by means of the density matrix

$$\rho = \psi\psi^\dagger = \psi_0\psi_0^\dagger,$$

which is defined on the ray.

These density operators are “pure” and of rank one:

$$\rho^\dagger = \rho; \quad \rho > 0; \quad \text{tr } \rho = 1; \quad \rho^2 = \rho.$$

There is a probability addition of the density operators. If ρ_1 and ρ_2 are two pure density matrices

$$\rho = \cos^2 \Theta \rho_1 + \sin^2 \Theta \rho_2$$

also is a density matrix; but it is not pure, but mixed. These are the interior points of the convex set of density operators. Hence

So

$$\rho^2 \neq \rho, \quad \rho - \rho^2 > 0.$$

Can we have an additional operation of constructing a **pure** density operator from the **sum** of the two density operators ρ_1, ρ_2 ? We must have such a construction since the superposition principle of quantum mechanics[1] tells us how to add two state vectors ψ_1 and ψ_2 to form a superposed state vector

$$\psi = \left\{ \cos \Theta \psi_1 + e^{i\varphi} \sin \Theta \psi_2 \right\}$$

which is normalized if ψ_1, ψ_2 are orthonormal. For nonorthogonal states, we have to use the normalization factor

$$(1 + \sin 2\Theta \cos \varphi |\langle \psi_1 | \psi_2 \rangle|)^{-1/2}.$$

where φ includes the phase of $\langle \psi_1 | \psi_2 \rangle$ along with φ^1 . In terms of density operators, the φ -addition law which we introduce is

$$\begin{aligned}\rho(x, y, \varphi) &= \{\cos \Theta \psi_1(x) + e^{i\varphi} \sin \Theta \psi_2(x)\} \{\cos \Theta \psi_1^\dagger(y) + e^{i\varphi} \sin \Theta \psi_2^\dagger(y)\} \\ &= \cos^2 \Theta \rho_1(x, y) + \sin^2 \Theta \rho_2(x, y) + \sin 2\Theta \cos \varphi \rho_{12}(x, y, \varphi).\end{aligned}$$

In terms of the Wigner function[2], we get a φ -addition law which has an interference term proportional to $\sin 2\Theta \cos \varphi$, i.e.

$$W(q, p) = \cos^2 \Theta W_1(q, p) + \sin^2 \Theta W_2(q, p) + \sin 2\Theta \cos \varphi I_{12}(q, p, \varphi)$$

where I_{12} is the (generalized) Wigner function corresponding to the operator having the structure of the root square of a convolution of the product of the two density matrices, which we symbolically denote as

$$I_{12} \rightarrow \sqrt{\rho_1 \rho_2}.$$

There is thus a one-parameter addition law of density operators and of Wigner functions with probabilities $\cos^2 \Theta$ and $\sin^2 \Theta$ and with extra interference term. Note that this $W(q, p)$ is pure and satisfies the purity criterion

$$\int \int dp dq \{W(q, p)\}^2 = \frac{1}{2\pi}.$$

For quantum tomography[3], also we can construct superposition of tomograms using a one parameter addition law:

$$\Phi(\lambda, \mu; x) = \cos^2 \Theta \Phi_1(\lambda, \mu; x) + \sin^2 \Theta \Phi_2(\lambda, \mu; x) + \sin 2\Theta \cos \varphi \Phi_{12}(\lambda, \mu; x).$$

Here tomograms Φ_1, Φ_2 determine the probability density of quadrature $x = \lambda q + \mu p$ in the pure states ψ_1 and ψ_2 . The tomogram Φ_{12} corresponds to the interference term I_{12} .

The passage from the impure density operator $\cos^2 \Theta \rho_1 + \sin^2 \Theta \rho_2$ to the pure φ -dependent addition may be called purification. Note that the purification introduces the relative phase φ which was not in ρ_1 or ρ_2 .

The density operator of a composite system AB with subsystems A and B may be chosen pure or impure. For a pure density operator ρ_{AB} , one can get the density operator ρ_A and ρ_B by the partial trace operation

$$\rho_A = \text{tr}_B (\rho_{AB}); \quad \rho_B = \text{tr}_A (\rho_{AB}).$$

It is not necessary that A and B have the same dimensionality. Unless ρ_{AB} is a direct product of pure states of A and B , a pure ρ_{AB} yields impure ρ_A and ρ_B . But they will have the same rank R and the same nonnegative eigenvalues which sum up to unity. The density operator

$$\rho'_{AB} = \rho_A \otimes \rho_B \neq \rho_{AB}$$

is impure. Thus, the whole is greater than the parts: there is additional information in ρ_{AB} . These are the “entanglement” terms[4].

We purify the product ρ'_{AB} in the same way that we used before for the mixture of two density operators ρ_1 and ρ_2 . Here we have n such pure states mixed together and need $(n-1)$ phase angles $\varphi_1 = 0, \varphi_2, \dots, \varphi_n$. The diagonal form of ρ'_{AB} has only n nonzero diagonal elements. We need to introduce the offdiagonal elements

$$\sqrt{\lambda_j \lambda_k} e^{i(\varphi_j - \varphi_k)}$$

in the (j, k) location. Note that while we have $n(n-1)/2$ offdiagonal terms, there are only $(n-1)$ phases φ_j .

The purification of the density matrix ρ'_{AB} we call as the φ - multiplication law of the density matrices ρ_A and ρ_B .

While purification of an impure density addition is dependent on one phase angle, the form of the entanglement is constructed depending on $R - 1$ phase angles. These have to be obtained from other considerations.

The same kind of φ -addition law and φ -multiplication law holds for other representatives like the quantum tomograms, the diagonal coherent state distribution function in quantum optics[5] and the Husimi - Kano[6] density of coherent state projection operators. We expect to return to this discussion elsewhere.

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